

ASYMPTOTIC SAFETY IN QUANTUM EINSTEIN GRAVITY: NONPERTURBATIVE RENORMALIZABILITY AND FRACTAL SPACETIME STRUCTURE¹

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Abstract

The asymptotic safety scenario of Quantum Einstein Gravity, the quantum field theory of the spacetime metric, is reviewed and it is argued that the theory is likely to be nonperturbatively renormalizable. It is also shown that asymptotic safety implies that spacetime is a fractal in general, with a fractal dimension of 2 on sub-Planckian length scales.

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1 Introduction

Quantized General Relativity, based upon the Einstein-Hilbert action

$$S_{\text{EH}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \{-R + 2\Lambda\} , \quad (1.1)$$

is well known to be perturbatively nonrenormalizable. This has led to the widespread believe that a straightforward quantization of the metric degrees of freedom cannot lead to a mathematically consistent and predictive *fundamental* theory valid down to arbitrarily small spacetime distances. Einstein gravity was rather considered merely an *effective* theory whose range of applicability is limited to a phenomenological description of gravitational effects at distances much larger than the Planck length.

In particle physics one usually considers a theory fundamental if it is perturbatively renormalizable. The virtue of such models is that one can “hide” their infinities in only finitely many basic parameters (masses, gauge couplings, etc.) which are intrinsically undetermined within the theory and whose value must be taken from the experiment. All other couplings are then well-defined computable functions of those few parameters. In nonrenormalizable effective theories, on the other hand, the divergence structure is such that increasing orders of the loop expansion require an increasing number of new counter terms and, as a consequence, of undetermined free parameters. Typically, at high energies, all these unknown parameters enter on an equal footing so that the theory loses its predictive power.

However, there are examples of field theories which do “exist” as fundamental theories despite their perturbative nonrenormalizability [1, 2]. These models are “nonperturbatively renormalizable” along the lines of Wilson’s modern formulation of renormalization theory [1]. They are constructed by performing the limit of infinite ultraviolet cutoff (“continuum limit”) at a non-Gaussian renormalization group fixed point g_{*i} in the space $\{g_i\}$ of all (dimensionless, essential) couplings g_i which parametrize a general action functional. This construction has to be contrasted with the standard perturbative renormalization which, at least implicitly, is based upon the Gaussian fixed point at which all couplings

vanish, $g_{*i} = 0$ [3, 4].

2 Asymptotic safety

In his “asymptotic safety” scenario Weinberg [5] has put forward the idea that, perhaps, a quantum field theory of gravity can be constructed nonperturbatively by invoking a non-Gaussian ultraviolet (UV) fixed point ($g_{*i} \neq 0$). The resulting theory would be “asymptotically safe” in the sense that at high energies unphysical singularities are likely to be absent.

The arena in which the idea is formulated is the so-called “theory space”. By definition, it is the space of all action functionals $A[\cdot]$ which depend on a given set of fields and are invariant under certain symmetries. Hence the theory space $\{A[\cdot]\}$ is fixed once the field contents and the symmetries are fixed. The infinitely many generalized couplings g_i needed to parametrize a general action functional are local coordinates on theory space. In gravity one deals with functionals $A[g_{\mu\nu}, \dots]$ which are required to depend on the metric in a diffeomorphism invariant way. (The dots represent matter fields and possibly background fields introduced for technical convenience.) Theory space carries a crucial geometric structure, namely a vector field which encodes the effect of a Kadanoff-Wilson-type block spin or “coarse graining” procedure, suitably reformulated in the continuum. The components β_i of this vector field are the beta-functions of the couplings g_i . They describe the dependence of $g_i \equiv g_i(k)$ on the coarse graining scale k :

$$k \partial_k g_i = \beta_i(g_1, g_2, \dots) \quad (2.1)$$

By definition, k is taken to be a mass scale. Roughly speaking the running couplings $g_i(k)$ describe the dynamics of field averages, the averaging volume having a linear extension of the order $1/k$. The $g_i(k)$ ’s should be thought of as parametrizing a running action functional $\Gamma_k[g_{\mu\nu}, \dots]$. By definition, the renormalization group (RG) trajectories, i.e. the solutions to the “exact renormalization group equation” (2.1) are the integral curves of the vector field $\vec{\beta} \equiv (\beta_i)$ defining the “RG flow”.

The asymptotic safety scenario assumes that $\vec{\beta}$ has a zero at a point with coordinates g_{*i} not all of which are zero. Given such a non-Gaussian fixed point (NGFP) of the RG flow one defines its UV critical surface \mathcal{S}_{UV} to consist of all points of theory space which are attracted into it in the limit $k \rightarrow \infty$. (Note that increasing k amounts to going in the direction *opposite* to the natural coarse graining flow.) The dimensionality $\dim(\mathcal{S}_{UV}) \equiv \Delta_{UV}$ is given by the number of attractive (for *increasing* cutoff k) directions in the space of couplings. The linearized flow near the fixed point is governed by the Jacobi matrix $\mathbf{B} = (B_{ij})$, $B_{ij} \equiv \partial_j \beta_i(g_*)$:

$$k \partial_k g_i(k) = \sum_j B_{ij} (g_j(k) - g_{*j}) . \quad (2.2)$$

The general solution to this equation reads

$$g_i(k) = g_{*i} + \sum_I C_I V_i^I \left(\frac{k_0}{k} \right)^{\theta_I} \quad (2.3)$$

where the V^I 's are the right-eigenvectors of \mathbf{B} with (complex) eigenvalues $-\theta_I$. Furthermore, k_0 is a fixed reference scale, and the C_I 's are constants of integration. If $g_i(k)$ is to approach g_{*i} in the infinite cutoff limit $k \rightarrow \infty$ we must set $C_I = 0$ for all I with $\text{Re } \theta_I < 0$. Hence the dimensionality Δ_{UV} equals the number of \mathbf{B} -eigenvalues with a negative real part, i.e. the number of θ_I 's with a positive real part.

A specific quantum field theory is defined by a RG trajectory which exists globally, i.e. is well behaved all the way down from “ $k = \infty$ ” in the UV to $k = 0$ in the IR. The key idea of asymptotic safety is to base the theory upon one of the trajectories running inside the hypersurface \mathcal{S}_{UV} since these trajectories are manifestly well-behaved and free from fatal singularities (blowing up couplings, etc.) in the large- k limit. Moreover, a theory based upon a trajectory inside \mathcal{S}_{UV} can be predictive, the problem of an increasing number of counter terms and undetermined parameters which plagues effective theory does not arise.

In fact, in order to select a specific quantum theory we have to fix Δ_{UV} free parameters which are not predicted by the theory and must be taken from experiment. When we

lower the cutoff, only Δ_{UV} parameters in the initial action are “relevant”, and fixing these parameters amounts to picking a specific trajectory on \mathcal{S}_{UV} ; the remaining “irrelevant” parameters are all attracted towards \mathcal{S}_{UV} automatically. Therefore the theory has the more predictive power the smaller is the dimensionality of \mathcal{S}_{UV} , i.e. the fewer UV attractive eigendirections the non-Gaussian fixed point has. If $\Delta_{\text{UV}} < \infty$, the quantum field theory thus constructed is as predictive as a perturbatively renormalizable model with Δ_{UV} “renormalizable couplings”, i.e. couplings relevant at the Gaussian fixed point.

It is plausible that \mathcal{S}_{UV} is indeed finite dimensional. If the dimensionless g_i ’s arise as $g_i(k) = k^{-d_i} \bar{g}_i(k)$ by rescaling (with the cutoff k) the original couplings \bar{g}_i with mass dimensions d_i , then $\beta_i = -d_i g_i + \dots$ and $B_{ij} = -d_i \delta_{ij} + \dots$ where the dots stand for the quantum corrections. Ignoring them, $\theta_i = d_i + \dots$, and Δ_{UV} equals the number of positive d_i ’s. Since adding derivatives or powers of fields to a monomial in the action always lowers d_i , there can be at most a finite number of positive d_i ’s and, therefore, of negative eigenvalues of \mathbf{B} . Thus, barring the presumably rather exotic possibility that the quantum corrections change the signs of infinitely many elements in \mathbf{B} , the dimensionality of \mathcal{S}_{UV} is finite [5].

We emphasize that in general the UV fixed point on which the above construction is based, if it exists, has no reason to be of the simple Einstein-Hilbert form (1.1). The initial point of the RG trajectory $\Gamma_{k \rightarrow \infty}$ is expected to contain many more invariants, both local (curvature polynomials) and nonlocal ones. For this reason the asymptotic safety scenario is *not* a quantization of General Relativity, and it cannot be compared in this respect to the loop quantum gravity approach, for instance. In a conventional field theory setting the functional $\Gamma_{k \rightarrow \infty}$ corresponds to the bare (or “classical”) action S which usually can be chosen (almost) freely. It is one of the many attractive features of the asymptotic safety scenario that the bare action is fixed by the theory itself and actually can be *computed*, namely by searching for zeros of $\vec{\beta}$. In this respect it has, almost by construction, a degree of predictivity which cannot be reached by any scheme trying to quantize a given classical action.

3 RG flow of the effective average action

During the past few years, the asymptotic safety scenario in Quantum Einstein Gravity (QEG) has been mostly investigated in the framework of the effective average action [6]-[21], [4], a specific formulation of the Wilsonian RG which originally was developed for theories in flat space [22, 23, 24] and has been first applied to gravity in [6].

Quite generally, the effective average action Γ_k is a coarse grained free energy functional that describes the behavior of the theory at the mass scale k . It contains the quantum effects of all fluctuations of the dynamical variables with momenta larger than k , but not of those with momenta smaller than k . As k is decreased, an increasing number of degrees of freedom is integrated out. The method thus complies, at an intuitive level, with the coarse graining picture of the previous section. The successive averaging of the fluctuation variable is achieved by a k -dependent IR cutoff term $\Delta_k S$ which is added to the classical action in the standard Euclidean functional integral. This term gives a momentum dependent mass square $\mathcal{R}_k(p^2)$ to the field modes with momentum p . It is designed to vanish if $p^2 \gg k^2$, but suppresses the contributions of the modes with $p^2 < k^2$ to the path integral. When regarded as a function of k , Γ_k describes a curve in theory space that interpolates between the classical action $S = \Gamma_{k \rightarrow \infty}$ and the conventional effective action $\Gamma = \Gamma_{k=0}$. The change of Γ_k induced by an infinitesimal change of k is described by a functional differential equation, the exact RG equation. In a symbolic notation it reads

$$k \partial_k \Gamma_k = \frac{1}{2} \text{STr} \left[\left(\Gamma_k^{(2)} + \mathcal{R}_k \right)^{-1} k \partial_k \mathcal{R}_k \right]. \quad (3.1)$$

For a detailed discussion of this equation we must refer to the literature [6]. Suffice it to say that, expanding $\Gamma_k[g_{\mu\nu}, \dots]$ in terms of diffeomorphism invariant field monomials $I_i[g_{\mu\nu}, \dots]$ with coefficients $g_i(k)$, eq. (3.1) assumes the component form (2.1).

In general it is impossible to find exact solutions to eq. (3.1) and we are forced to rely upon approximations. A powerful nonperturbative approximation scheme is the truncation of theory space where the RG flow is projected onto a finite-dimensional subspace.

In practice one makes an ansatz for Γ_k that comprises only a few couplings and inserts it into the RG equation. This leads to a, now finite, set of coupled differential equations of the form (2.1).

The simplest approximation one might try is the “Einstein-Hilbert truncation” [6, 8] defined by the ansatz

$$\Gamma_k[g_{\mu\nu}] = (16\pi G_k)^{-1} \int d^d x \sqrt{g} \{-R(g) + 2\bar{\lambda}_k\} \quad (3.2)$$

It applies to a d -dimensional Euclidean spacetime and involves only the cosmological constant $\bar{\lambda}_k$ and the Newton constant G_k as running parameters. Inserting (3.2) into the RG equation (3.1) one obtains a set of two β -functions (β_λ, β_g) for the dimensionless cosmological constant $\lambda_k \equiv k^{-2}\bar{\lambda}_k$ and the dimensionless Newton constant $g_k \equiv k^{d-2}G_k$, respectively. They describe a two-dimensional RG flow on the plane with coordinates $g_1 \equiv \lambda$ and $g_2 \equiv g$. At a fixed point (λ_*, g_*) , both β -functions vanish simultaneously. In the Einstein-Hilbert truncation there exists both a trivial Gaussian fixed point (GFP) at $\lambda_* = g_* = 0$ and, quite remarkably, also a UV attractive NGFP at $(\lambda_*, g_*) \neq (0, 0)$.

In Fig. 1 we show part of the g - λ theory space and the corresponding RG flow for $d = 4$. The trajectories are obtained by numerically integrating the differential equations $k \partial_k \lambda = \beta_\lambda(\lambda, g)$ and $k \partial_k g = \beta_g(\lambda, g)$. The arrows point in the direction of increasing coarse graining, i.e. from the UV towards the IR. We observe that three types of trajectories emanate from the NGFP: those of Type Ia (Type IIIa) run towards negative (positive) cosmological constants, while the “separatrix”, the unique trajectory (of Type IIa) crossing over from the NGFP to the GFP, has a vanishing cosmological constant in the IR. The flow is defined on the half-plane $\lambda < 1/2$ only; it cannot be continued beyond $\lambda = 1/2$ as the β -functions become singular there. In fact, the Type IIIa-trajectories cannot be integrated down to $k = 0$ within the Einstein-Hilbert approximation. They terminate at a non-zero k_{term} where they run into the $\lambda = 1/2$ -singularity. Near k_{term} a more general truncation is needed in order to continue the flow.

In Weinberg’s original paper [5] the asymptotic safety idea was tested in $d = 2 + \epsilon$

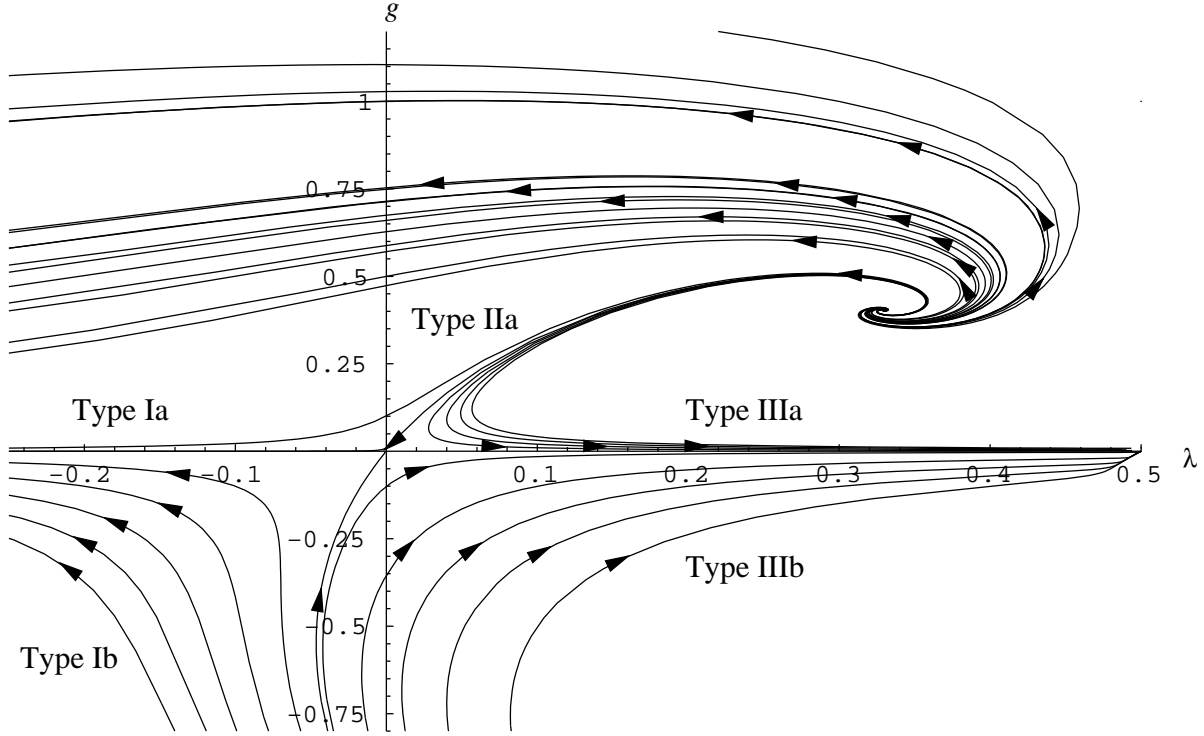


Figure 1: Part of theory space of the Einstein-Hilbert truncation with its RG flow. The arrows point in the direction of decreasing values of k . The flow is dominated by a non-Gaussian fixed point in the first quadrant and a trivial one at the origin. (From [9].)

dimensions where $0 < \epsilon \ll 1$ was chosen so that the β -functions (actually β_g only) could be found by an ϵ -expansion. Before the advent of the exact RG equations no practical tool was known which would have allowed a nonperturbative calculation of the β -functions in the physically interesting case of $d = 4$ spacetime dimensions. However, as we saw above, the effective average action in the Einstein-Hilbert approximation does indeed predict the existence of a NGFP in a nonperturbative setting. It was first analyzed in [13, 8, 9], and also first investigations of its possible role in black hole physics [25] and cosmology [26, 27] were performed already.

The detailed analyses of refs. [8, 9] demonstrated that the NGFP found has all the properties necessary for asymptotic safety. In particular one has a pair of complex conjugate critical exponents $\theta' \pm i\theta''$ with $\theta' > 0$, implying that the NGFP, for $k \rightarrow \infty$, attracts all trajectories in the half-plane $g > 0$. (The lower half-plane $g < 0$ is unphysical prob-

ably since it corresponds to a negative Newton constant.) Because of the nonvanishing imaginary part $\theta'' \neq 0$, all trajectories spiral around the NGFP before hitting it.

The question of crucial importance is whether the fixed point predicted by the Einstein-Hilbert truncation actually approximates a fixed point in the exact theory, or whether it is an artifact of the truncation. In refs. [8, 10, 9] evidence was found which, in our opinion, strongly supports the hypothesis that there does indeed exist a non-Gaussian fixed point in the exact 4-dimensional theory, with exactly the properties required for the asymptotic safety scenario. In these investigations the reliability of the Einstein-Hilbert truncation was tested both by analyzing the cutoff scheme dependence within this truncation [8, 9] and by generalizing the truncation ansatz itself [10]. The idea behind the first method is as follows.

The cutoff operator $\mathcal{R}_k(p^2)$ is specified by a matrix in field space and a “shape function” $R^{(0)}(p^2/k^2)$ which describes the details of how the modes get suppressed in the IR when p^2 drops below k^2 . We checked the cutoff scheme dependence of the various quantities of interest both by looking at their dependence on the function $R^{(0)}$ and comparing two different matrix structures. Universal quantities are particularly important in this respect because, by definition, they are strictly cutoff scheme independent in the exact theory. Any truncation leads to a residual scheme dependence of these quantities, however. Its magnitude is a natural indicator for the quality of the truncation [28]. Typical examples of universal quantities are the critical exponents θ_I . The existence or nonexistence of a fixed point is also a universal, scheme independent feature, but its precise location in parameter space is scheme dependent. Nevertheless it can be shown that, in $d = 4$, the product $g_*\lambda_*$ must be universal [8] while g_* and λ_* separately are not.

The detailed numerical analysis of the Einstein-Hilbert RG flow near the NGFP [8, 9] shows that the universal quantities, in particular the product $g_*\lambda_*$, are indeed scheme independent at a quite impressive level of accuracy. As the many numerical “miracles” which lead to the almost perfect cancellation of the $R^{(0)}$ -dependence would have no reason to occur if there was not a fixed point in the exact theory as an organizing principle, the

results of this analysis can be considered strong evidence in favor of a fixed point in the exact, un-truncated theory.

The ultimate justification of any truncation is that when one adds further terms to it its physical predictions do not change significantly any more. As a first step towards testing the stability of the Einstein-Hilbert truncation against the inclusion of other invariants [10] we took a (curvature)²-term into account:

$$\Gamma_k[g_{\mu\nu}] = \int d^d x \sqrt{g} \{ (16\pi G_k)^{-1} [-R(g) + 2\bar{\lambda}_k] + \bar{\beta}_k R^2(g) \} \quad (3.3)$$

Inserting (3.3) into the functional RG equation yields a set of β -functions $(\beta_\lambda, \beta_g, \beta_\beta)$ for the dimensionless couplings λ_k , g_k and $\beta_k \equiv k^{4-d}\bar{\beta}_k$. They describe the RG flow on the three-dimensional λ - g - β -space. Despite the extreme algebraic complexity of the three β -functions it was possible to show [10, 11, 12] that they, too, admit a NGFP $(\lambda_*, g_*, \beta_*)$ with exactly the properties needed for asymptotic safety. In particular it turned out to be UV attractive in all three directions. The value of β_* is extremely tiny, and close to the NGFP the projection of the 3-dimensional flow onto the λ - g -subspace is very well described by the Einstein-Hilbert truncation which ignores the third direction from the outset. The λ_* - and g_* -values and the critical exponents related to the flow in the λ - g -subspace, as predicted by the 3-dimensional truncation, agree almost perfectly with those from the Einstein-Hilbert approximation. Analyzing the scheme dependence of the universal quantities one finds again a highly remarkable $R^{(0)}$ -independence – which is truly amazing if one visualizes the huge amount of nontrivial numerical compensations and cancellations among several dozens of $R^{(0)}$ -dependent terms which is necessary to make $g_* \lambda_*$, say, approximately independent of the shape function $R^{(0)}$.

On the basis of these results we believe that the non-Gaussian fixed point occurring in the Einstein-Hilbert truncation is very unlikely to be an artifact of this truncation but rather may be considered the projection of a NGFP in the exact theory. The fixed point and all its qualitative properties are stable against variations of the cutoff and the inclusion of a further invariant in the truncation. It is particularly remarkable that

within the scheme dependence the additional R^2 -term has essentially no impact on the fixed point. These are certainly very nontrivial indications supporting the conjecture that 4-dimensional QEG indeed possesses a RG fixed point with the properties needed for its nonperturbative renormalizability.

This view is further supported by two conceptually independent investigations. In ref. [19] a proper time renormalization group equation rather than the flow equation of the average action has been used, and again a suitable NGFP was found. This framework is conceptually somewhat simpler than that of the effective average action; it amounts to an RG-improved 1-loop calculation with an IR cutoff. Furthermore, in refs. [29] the functional integral over the subsector of metrics admitting two Killing vectors has been performed *exactly*, and again a NGFP was found, this time in a setting and an approximation which is *very* different from that of the truncated Γ_k -flows. As for the inclusion of matter fields, both in the average action [14, 15, 16, 20] and the symmetry reduction approach [29] a suitable NGFP has been established for a broad class of matter systems.

4 Scale dependent metrics and the resolution function $\ell(k)$

In the following we take the existence of a suitable NGFP on the full theory space for granted and explore some of the properties of asymptotic safety, in particular we try to gain some understanding of what a “quantum spacetime” is like. Unless stated otherwise we consider pure Euclidean gravity in $d = 4$.

The running effective average action $\Gamma_k[g_{\mu\nu}]$ defines an infinite set of effective field theories, valid near the scale k which we may vary between $k = 0$ and $k = \infty$. Intuitively speaking, the solution $\langle g_{\mu\nu} \rangle_k$ of the scale dependent field equation

$$\frac{\delta \Gamma_k}{\delta g_{\mu\nu}(x)}[\langle g \rangle_k] = 0 \tag{4.1}$$

can be interpreted as the metric averaged over (Euclidean) spacetime volumes of a linear extension ℓ which typically is of the order of $1/k$. Knowing the scale dependence of Γ_k , i.e. the renormalization group trajectory $k \mapsto \Gamma_k$, we can derive the running effective Einstein equations (4.1) for any k and, after fixing appropriate boundary conditions and symmetry requirements, follow their solution $\langle g_{\mu\nu} \rangle_k$ from $k = \infty$ to $k = 0$.

The infinitely many equations of (4.1), one for each scale k , are valid *simultaneously*. They all refer *to the same* physical system, the “quantum spacetime”. They describe its effective metric structure on different length scales. An observer using a “microscope” with a resolution $\approx k^{-1}$ will perceive the universe to be a Riemannian manifold with metric $\langle g_{\mu\nu} \rangle_k$. At every fixed k , $\langle g_{\mu\nu} \rangle_k$ is a smooth classical metric. But since the quantum spacetime is characterized by the infinity of metrics $\{\langle g_{\mu\nu} \rangle_k | k = 0, \dots, \infty\}$ it can acquire very nonclassical and in particular fractal features. In fact, every proper distance calculated from $\langle g_{\mu\nu} \rangle_k$ is unavoidably scale dependent. This phenomenon is familiar from fractal geometry, a famous example being the coast line of England whose length depends on the size of the yardstick used to measure it [30].

Let us describe more precisely what it means to “average” over Euclidean spacetime volumes. The quantity we can freely tune is the IR cutoff scale k , and the “resolving power” of the microscope, henceforth denoted ℓ , is an a priori unknown function of k . (In flat space, $\ell \approx \pi/k$.) In order to understand the relationship between ℓ and k we must recall some more steps from the construction of $\Gamma_k[g_{\mu\nu}]$ in ref. [6].

The effective average action is obtained by introducing an IR cutoff into the path-integral over all metrics, gauge fixed by means of a background gauge fixing condition. Even without a cutoff the resulting effective action $\Gamma[g_{\mu\nu}; \bar{g}_{\mu\nu}]$ depends on two metrics, the expectation value of the quantum field, $g_{\mu\nu}$, and the background field $\bar{g}_{\mu\nu}$. This is a standard technique, and it is well known [31] that the functional $\Gamma[g_{\mu\nu}] \equiv \Gamma[g_{\mu\nu}; \bar{g}_{\mu\nu} = g_{\mu\nu}]$ obtained by equating the two metrics can be used to generate the 1PI Green’s functions of the theory.

(We emphasize, however, that the average action method is manifestly *background*

independent despite the temporary use of $\bar{g}_{\mu\nu}$ at an intermediate level. At no stage in the derivation of the β -functions it is necessary to assign a concrete metric to $\bar{g}_{\mu\nu}$, such as $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ in standard perturbation theory, say. The RG flow, i.e. the vector field $\vec{\beta}$, on the theory space of diffeomorphism invariant action functionals depending on $g_{\mu\nu}$ and $\bar{g}_{\mu\nu}$ is a highly universal object: it neither depends on any specific metric, nor on any specific action.)

The IR cutoff of the average action is implemented by first expressing the functional integral over all metric fluctuations in terms of eigenmodes of \bar{D}^2 , the covariant Laplacian formed with the aid of the background metric $\bar{g}_{\mu\nu}$. Then the suppression term $\Delta_k S$ is introduced which damps the contribution of all $-\bar{D}^2$ -modes with eigenvalues smaller than k^2 . Coupling the dynamical fields to sources and Legendre-transforming leads to the scale dependent functional $\Gamma_k[g_{\mu\nu}; \bar{g}_{\mu\nu}]$, and the action with one argument again obtains by equating the two metrics: $\Gamma_k[g_{\mu\nu}] \equiv \Gamma_k[g_{\mu\nu}; \bar{g}_{\mu\nu} = g_{\mu\nu}]$. It is this action which appears in (4.1). Because of the identification of the two metrics we see that, in a sense, it is the eigenmodes of $\bar{D}^2 = D^2$, constructed from the argument of $\Gamma_k[g]$, which are cut off at k^2 .

This last observation is essential for the following algorithm [23, 32] for the reconstruction of the averaging scale ℓ from the cutoff k . The input data is the set of metrics characterizing a quantum manifold, $\{\langle g_{\mu\nu} \rangle_k\}$. The idea is to deduce the relation $\ell = \ell(k)$ from the spectral properties of the scale dependent Laplacian $\Delta(k) \equiv D^2(\langle g_{\mu\nu} \rangle_k)$ built with the solution of the effective field equation. More precisely, for every fixed value of k , one solves the eigenvalue problem of $-\Delta(k)$ and studies the properties of the special eigenfunctions whose eigenvalue is k^2 , or nearest to k^2 in the case of a discrete spectrum. We shall refer to an eigenmode of $-\Delta(k)$ whose eigenvalue is (approximately) the square of the cutoff k as a “cutoff mode” (COM) and denote the set of all COMs by $\text{COM}(k)$.

If we ignore the k -dependence of $\Delta(k)$ for a moment (as it would be appropriate for matter theories in flat space) the COMs are, for a sharp cutoff, precisely the last modes integrated out when lowering the cutoff, since the suppression term in the path integral cuts out all modes of the metric fluctuation with eigenvalue smaller than k^2 .

For a non-gauge theory in flat space the coarse graining or averaging of fields is a well defined procedure, based upon ordinary Fourier analysis, and one finds that in this case the length ℓ is essentially the wave length of the last modes integrated out, the COMs.

This observation motivates the following definition of ℓ in quantum gravity. We determine the COMs of $-\Delta(k)$, analyze how fast these eigenfunctions vary on spacetime, and read off a typical coordinate distance Δx^μ characterizing the scale on which they vary. For an oscillatory COM, for example, Δx^μ would correspond to an oscillation period. (In general there is a certain freedom in the precise identification of the Δx^μ belonging to a specific cutoff mode. This ambiguity can be resolved by refining the definition of Δx^μ on a case-by-case basis only.) Finally we use the metric $\langle g_{\mu\nu} \rangle_k$ itself in order to convert Δx^μ to a proper length. This proper length, by definition, is ℓ . Repeating the above steps for all values of k , we end up with a function $\ell = \ell(k)$. In general one will find that ℓ depends on the position on the manifold as well as on the direction of Δx^μ .

Applying the above algorithm on a non-dynamical flat spacetime one recovers the expected result $\ell(k) = \pi/k$. In ref. [32] a specific example of a QEG spacetime has been constructed, the quantum S^4 , which is an ordinary 4-sphere at every fixed scale, with a k -dependent radius, though. In this case, too, the resolution function was found to be $\ell(k) = \pi/k$.

Thus the construction and interpretation of a QEG spacetime proceeds, in a nutshell, as follows. We start from a fixed RG trajectory $k \mapsto \Gamma_k$, derive its effective field equations at each k , and solve them. The resulting quantum mechanical counterpart of a classical spacetime is equipped with the infinity of Riemannian metrics $\{\langle g_{\mu\nu} \rangle_k | k = 0, \dots, \infty\}$ where the parameter k is only a book keeping device a priori. It can be given a physical interpretation by relating it to the COM length scale characterizing the averaging procedure: One constructs the Laplacian $-D^2(\langle g_{\mu\nu} \rangle_k)$, diagonalizes it, looks how rapidly its k^2 -eigenfunction varies, and “measures” the length of typical variations with the metric $\langle g_{\mu\nu} \rangle_k$ itself. In the ideal case one can solve the resulting $\ell = \ell(k)$ for $k = k(\ell)$ and reinterpret the metric $\langle g_{\mu\nu} \rangle_k$ as referring to a microscope with a known position and direction

dependent resolving power ℓ . The price we have to pay for the background independence is that we cannot freely choose ℓ directly but rather k only.

5 Microscopic structure of the QEG spacetimes

One of the intriguing conclusions we reached in refs. [8, 10] was that the QEG spacetimes are fractals and that their effective dimensionality is scale dependent. It equals 4 at macroscopic distances ($\ell \gg \ell_{\text{Pl}}$) but, near $\ell \approx \ell_{\text{Pl}}$, it gets dynamically reduced to the value 2. For $\ell \ll \ell_{\text{Pl}}$ spacetime is, in a precise sense [8], a 2-dimensional fractal.

In ref. [26] the specific form of the graviton propagator on this fractal was applied in a cosmological context. It was argued that it gives rise to a Harrison-Zeldovich spectrum of primordial geometry fluctuations, perhaps responsible for the CMBR spectrum observed today. (In refs. [25, 26, 27], [33]-[38] various types of “RG improvements” were used to explore possible physical manifestations of the scale dependence of the gravitational parameters.)

A priori there exist several plausible definitions of a fractal dimensionality of spacetime. In our original argument [8] we used the one based upon the anomalous dimension η_N at the NGFP. We shall review this argument in the rest of this section. Then, in Section 6, we evaluate the spectral dimension for the QEG spacetimes [39] and demonstrate that it displays the same dimensional reduction $4 \rightarrow 2$ as the one based upon η_N . The spectral dimension has also been determined in Monte Carlo simulations of causal (i.e. Lorentzian) dynamical triangulations [40]-[43] and it will be interesting to compare the results.

For simplicity we use the Einstein-Hilbert truncation to start with, and we consider spacetimes with classical dimensionality $d = 4$. The corresponding RG trajectories are shown in Fig. 1. For $k \rightarrow \infty$, all of them approach the NGFP (λ_*, g_*) so that the dimensionful quantities run according to

$$G_k \approx g_*/k^2 \quad , \quad \bar{\lambda}_k \approx \lambda_* k^2 \quad (5.1)$$

The behavior (5.1) is realized in the asymptotic scaling regime $k \gg m_{\text{Pl}}$. Near $k = m_{\text{Pl}}$ the trajectories cross over towards the GFP. Since we are interested only in the limiting cases of very small and very large distances the following caricature of a RG trajectory will be sufficient. We assume that G_k and $\bar{\lambda}_k$ behave as in (5.1) for $k \gg m_{\text{Pl}}$, and that they assume constant values for $k \ll m_{\text{Pl}}$. The precise interpolation between the two regimes could be obtained numerically [9] but will not be needed here.

The argument of ref. [10] concerning the fractal nature of the QEG spacetimes is as follows. Within the Einstein-Hilbert truncation of theory space, the effective field equations (4.1) happen to coincide with the ordinary Einstein equation, but with G_k and $\bar{\lambda}_k$ replacing the classical constants. Without matter,

$$R_{\mu\nu}(\langle g \rangle_k) = \bar{\lambda}_k \langle g_{\mu\nu} \rangle_k \quad (5.2)$$

Since in absence of dimensionful constants of integration $\bar{\lambda}_k$ is the only quantity in this equation which sets a scale, every solution to (5.2) has a typical radius of curvature $r_c(k) \propto 1/\sqrt{\bar{\lambda}_k}$. (For instance, the S^4 -solution has the radius $r_c = \sqrt{3/\bar{\lambda}_k}$.) If we want to explore the spacetime structure at a fixed length scale ℓ we should use the action $\Gamma_k[g_{\mu\nu}]$ at $k \approx \pi/\ell$ because with this functional a tree level analysis is sufficient to describe the essential physics at this scale, including the relevant quantum effects. Hence, when we observe the spacetime with a microscope of resolution ℓ , we will see an average radius of curvature given by $r_c(\ell) \equiv r_c(k = \pi/\ell)$. Once ℓ is smaller than the Planck length $\ell_{\text{Pl}} \equiv m_{\text{Pl}}^{-1}$ we are in the fixed point regime where $\bar{\lambda}_k \propto k^2$ so that $r_c(k) \propto 1/k$, or

$$r_c(\ell) \propto \ell \quad (5.3)$$

Thus, when we look at the structure of spacetime with a microscope of resolution $\ell \ll \ell_{\text{Pl}}$, the average radius of curvature which we measure is proportional to the resolution itself. If we want to probe finer details and decrease ℓ we automatically decrease r_c and hence *increase* the average curvature. Spacetime seems to be more strongly curved at small distances than at larger ones. The scale-free relation (5.3) suggests that at distances

below the Planck length the QEG spacetime is a special kind of fractal with a self-similar structure. It has no intrinsic scale because in the fractal regime, i.e. when the RG trajectory is still close to the NGFP, the parameters which usually set the scales of the gravitational interaction, G and $\bar{\lambda}$, are not yet “frozen out”. This happens only later on, somewhere half way between the NGFP and the GFP, at a scale of the order of m_{Pl} . Below this scale, G_k and $\bar{\lambda}_k$ stop running and, as a result, $r_c(k)$ becomes independent of k so that $r_c(\ell) = \text{const}$ for $\ell \gg \ell_{\text{Pl}}$. In this regime $\langle g_{\mu\nu} \rangle_k$ is k -independent, indicating that the macroscopic spacetime is describable by a single smooth Riemannian manifold.

The above argument made essential use of the proportionality $\ell \propto 1/k$. In the fixed point regime it follows trivially from the fact that there exist no other relevant dimensional parameters so that $1/k$ is the only length scale one can form. The algorithm for the determination of $\ell(k)$ described above yields the same answer.

It is easy to make the k -dependence of $\langle g_{\mu\nu} \rangle_k$ explicit. Picking an arbitrary reference scale k_0 we rewrite (5.2) as $[\bar{\lambda}_{k_0}/\bar{\lambda}_k] R^\mu{}_\nu(\langle g \rangle_k) = \bar{\lambda}_{k_0} \delta^\mu{}_\nu$. Since $R^\mu{}_\nu(cg) = c^{-1} R^\mu{}_\nu(g)$ for any constant $c > 0$, the average metric and its inverse scale as

$$\langle g_{\mu\nu}(x) \rangle_k = [\bar{\lambda}_{k_0}/\bar{\lambda}_k] \langle g_{\mu\nu}(x) \rangle_{k_0} \quad (5.4)$$

$$\langle g^{\mu\nu}(x) \rangle_k = [\bar{\lambda}_k/\bar{\lambda}_{k_0}] \langle g^{\mu\nu}(x) \rangle_{k_0} \quad (5.5)$$

These relations are valid provided the family of solutions considered exists for all scales between k_0 and k , and $\bar{\lambda}_k$ has the same sign always.

As we discussed in ref. [8] the QEG spacetime has an effective dimensionality which is k -dependent and hence noninteger in general. The discussion was based upon the anomalous dimension η_N of the operator $\int \sqrt{g} R$. It is defined as $\eta_N \equiv -k \partial_k \ln Z_{Nk}$ where $Z_{Nk} \propto 1/G_k$ is the wavefunction renormalization of the metric [6]. In a sense which we shall make more precise in a moment, the effective dimensionality of spacetime equals $4 + \eta_N$. The RG trajectories of the Einstein-Hilbert truncation (within its domain

of validity) have $\eta_N \approx 0$ for $k \rightarrow 0^2$ and $\eta_N \approx -2$ for $k \rightarrow \infty$, the smooth change by two units occuring near $k \approx m_{\text{Pl}}$. As a consequence, the effective dimensionality is 4 for $\ell \gg \ell_{\text{Pl}}$ and 2 for $\ell \ll \ell_{\text{Pl}}$.

The UV fixed point has an anomalous dimension $\eta \equiv \eta_N(\lambda_*, g_*) = -2$. We can use this information in order to determine the momentum dependence of the dressed graviton propagator for momenta $p^2 \gg m_{\text{Pl}}^2$. Expanding the Γ_k of (3.2) about flat space and omitting the standard tensor structures we find the inverse propagator $\tilde{\mathcal{G}}_k(p)^{-1} \propto Z_N(k) p^2$. The conventional dressed propagator $\tilde{\mathcal{G}}(p)$, the one contained in $\Gamma \equiv \Gamma_{k=0}$, obtains from the exact $\tilde{\mathcal{G}}_k$ by taking the limit $k \rightarrow 0$. For $p^2 > k^2 \gg m_{\text{Pl}}^2$ the actual cutoff scale is the physical momentum p^2 itself³ so that the k -evolution of $\tilde{\mathcal{G}}_k(p)$ stops at the threshold $k = \sqrt{p^2}$. Therefore

$$\tilde{\mathcal{G}}(p)^{-1} \propto Z_N(k = \sqrt{p^2}) p^2 \propto (p^2)^{1-\frac{\eta}{2}} \quad (5.6)$$

because $Z_N(k) \propto k^{-\eta}$ when $\eta \equiv -\partial_t \ln Z_N$ is (approximately) constant. In d dimensions, and for $\eta \neq 2 - d$, the Fourier transform of $\tilde{\mathcal{G}}(p) \propto 1/(p^2)^{1-\eta/2}$ yields the following propagator in position space:

$$\mathcal{G}(x; y) \propto \frac{1}{|x - y|^{d-2+\eta}} . \quad (5.7)$$

This form of the propagator is well known from the theory of critical phenomena, for instance. (In the latter case it applies to large distances.) Eq. (5.7) is not valid directly at the NGFP. For $d = 4$ and $\eta = -2$ the dressed propagator is $\tilde{\mathcal{G}}(p) = 1/p^4$ which has the following representation in position space:

$$\mathcal{G}(x; y) = -\frac{1}{8\pi^2} \ln(\mu |x - y|) . \quad (5.8)$$

Here μ is an arbitrary constant with the dimension of a mass. Obviously (5.8) has the same form as a $1/p^2$ -propagator in 2 dimensions.

²In the case of type IIIa trajectories [9, 37] the macroscopic k -value is still far above k_{term} , i.e. in the “GR regime” described in [37].

³See Section 1 of ref. [35] for a detailed discussion of “decoupling” phenomena of this kind.

Slightly away from the NGFP, before other physical scales intervene, the propagator is of the familiar type (5.7) which shows that the quantity η_N has the standard interpretation of an anomalous dimension in the sense that fluctuation effects modify the decay properties of \mathcal{G} so as to correspond to a spacetime of effective dimensionality $4 + \eta_N$.

Thus the properties of the RG trajectories imply the following “dimensional reduction”: Spacetime, probed by a “graviton” with $p^2 \ll m_{\text{Pl}}^2$ is 4-dimensional, but it appears to be 2-dimensional for a graviton with $p^2 \gg m_{\text{Pl}}^2$ [8].

It is interesting to note that in d classical dimensions, where the macroscopic spacetime is d -dimensional, the anomalous dimension at the fixed point is $\eta = 2 - d$. Therefore, for any d , the dimensionality of the fractal as implied by η_N is $d + \eta = 2$ [8, 10].

6 The spectral dimension

Next we turn to the spectral dimension \mathcal{D}_s of the QEG spacetimes. This particular definition of a fractal dimension is borrowed from the theory of diffusion processes on fractals [44] and is easily adapted to the quantum gravity context [45, 43]. In particular it has been used in the Monte Carlo studies mentioned above.

Let us study the diffusion of a scalar test particle on a d -dimensional classical Euclidean manifold with a fixed smooth metric $g_{\mu\nu}(x)$. The corresponding heat-kernel $K_g(x, x'; T)$ giving the probability for the particle to diffuse from x' to x during the fictitious diffusion time T satisfies the heat equation $\partial_T K_g(x, x'; T) = \Delta_g K_g(x, x'; T)$ where $\Delta_g \equiv D^2$ denotes the scalar Laplacian: $\Delta_g \phi \equiv g^{-1/2} \partial_\mu (g^{1/2} g^{\mu\nu} \partial_\nu \phi)$. The heat-kernel is a matrix element of the operator $\exp(T \Delta_g)$. In the random walk picture its trace per unit volume, $P_g(T) \equiv V^{-1} \int d^d x \sqrt{g(x)} K_g(x, x; T) \equiv V^{-1} \text{Tr} \exp(T \Delta_g)$, has the interpretation of an average return probability. (Here $V \equiv \int d^d x \sqrt{g}$ denotes the total volume.) It is well known that P_g possesses an asymptotic expansion (for $T \rightarrow 0$) of the form $P_g(T) = (4\pi T)^{-d/2} \sum_{n=0}^{\infty} A_n T^n$. For an infinite flat space, for instance, it reads $P_g(T) = (4\pi T)^{-d/2}$ for all T . Thus, knowing the function P_g , one can recover

the dimensionality of the target manifold as the T -independent logarithmic derivative $d = -2 d \ln P_g(T) / d \ln T$. This formula can also be used for curved spaces and spaces with finite volume V provided T is not taken too large [43].

In QEG where we functionally integrate over all metrics it is natural to replace $P_g(T)$ by its expectation value. Symbolically, $P(T) \equiv \langle P_\gamma(T) \rangle$ where $\gamma_{\mu\nu}$ denotes the microscopic metric (integration variable) and the expectation value is with respect to the ordinary path integral (without IR cutoff) containing the fixed point action. Given $P(T)$, we define the spectral dimension of the quantum spacetime in analogy with the classical formula:

$$\mathcal{D}_s = -2 \frac{d \ln P(T)}{d \ln T} \quad (6.1)$$

Let us now evaluate (6.1) using the average action method. The fictitious diffusion process takes place on a “manifold” which, at every fixed scale, is described by a smooth Riemannian metric $\langle g_{\mu\nu} \rangle_k$. While the situation appears to be classical at fixed k , non-classical features emerge in the regime with nontrivial RG running since there the metric depends on the scale at which the spacetime structure is probed.

The nonclassical features are encoded in the properties of the diffusion operator. Denoting the covariant Laplacians corresponding to the metrics $\langle g_{\mu\nu} \rangle_k$ and $\langle g_{\mu\nu} \rangle_{k_0}$ by $\Delta(k)$ and $\Delta(k_0)$, respectively, eqs. (5.4) and (5.5) imply that they are related by

$$\Delta(k) = [\bar{\lambda}_k / \bar{\lambda}_{k_0}] \Delta(k_0) \quad (6.2)$$

When $k, k_0 \gg m_{\text{Pl}}$ we have, for example,

$$\Delta(k) = (k/k_0)^2 \Delta(k_0) \quad (6.3)$$

Recalling that the average action Γ_k defines an effective field theory at the scale k we have that $\langle \mathcal{O}(\gamma_{\mu\nu}) \rangle \approx \mathcal{O}(\langle g_{\mu\nu} \rangle_k)$ if the operator \mathcal{O} involves typical covariant momenta of the order k and $\langle g_{\mu\nu} \rangle_k$ solves eq. (4.1). In the following we exploit this relationship for the RHS of the diffusion equation, $\mathcal{O} = \Delta_\gamma K_\gamma(x, x'; T)$. It is crucial here to correctly identify the relevant scale k .

If the diffusion process involves only a small interval of scales near k over which $\bar{\lambda}_k$ does not change much the corresponding heat equation must contain the $\Delta(k)$ for this specific, fixed value of k :

$$\partial_T K(x, x'; T) = \Delta(k) K(x, x'; T) \quad (6.4)$$

Denoting the eigenvalues of $-\Delta(k_0)$ by \mathcal{E}_n and the corresponding eigenfunctions by ϕ_n , this equation is solved by

$$K(x, x'; T) = \sum_n \phi_n(x) \phi_n(x') \exp\left(-F(k^2) \mathcal{E}_n T\right) \quad (6.5)$$

Here we introduced the convenient notation $F(k^2) \equiv \bar{\lambda}_k / \bar{\lambda}_{k_0}$. Knowing this propagation kernel we can time-evolve any initial probability distribution $p(x; 0)$ according to $p(x; T) = \int d^4x' \sqrt{g_0(x')} K(x, x'; T) p(x'; 0)$ with g_0 the determinant of $\langle g_{\mu\nu} \rangle_{k_0}$. If the initial distribution has an eigenfunction expansion of the form $p(x; 0) = \sum_n C_n \phi_n(x)$ we obtain

$$p(x; T) = \sum_n C_n \phi_n(x) \exp\left(-F(k^2) \mathcal{E}_n T\right) \quad (6.6)$$

If the C_n 's are significantly different from zero only for a single eigenvalue \mathcal{E}_N , we are dealing with a single-scale problem. In the usual spirit of effective field theories we would then identify $k^2 = \mathcal{E}_N$ as the relevant scale at which the running couplings are to be evaluated. However, in general the C_n 's are different from zero over a wide range of eigenvalues. In this case we face a multiscale problem where different modes ϕ_n probe the spacetime on different length scales.

If $\Delta(k_0)$ corresponds to flat space, say, the eigenfunctions $\phi_n \equiv \phi_p$ are plane waves with momentum p^μ , and they resolve structures on a length scale ℓ of order $\pi/|p|$. Hence, in terms of the eigenvalue $\mathcal{E}_n \equiv \mathcal{E}_p = p^2$ the resolution is $\ell \approx \pi/\sqrt{\mathcal{E}_n}$. This suggests that when the manifold is probed by a mode with eigenvalue \mathcal{E}_n it “sees” the metric $\langle g_{\mu\nu} \rangle_k$ for the scale $k = \sqrt{\mathcal{E}_n}$. Actually the identification $k = \sqrt{\mathcal{E}_n}$ is correct also for curved space since, in the construction of Γ_k , the parameter k is introduced precisely as a cutoff in the spectrum of the covariant Laplacian.

Therefore we conclude that under the spectral sum of (6.6) we must use the scale $k^2 = \mathcal{E}_n$ which depends explicitly on the resolving power of the corresponding mode. Likewise, in eq. (6.5), $F(k^2)$ is to be interpreted as $F(\mathcal{E}_n)$. Thus we obtain the traced propagation kernel

$$\begin{aligned} P(T) &= V^{-1} \sum_n \exp \left[-F(\mathcal{E}_n) \mathcal{E}_n T \right] \\ &= V^{-1} \text{Tr} \exp \left[F \left(-\Delta(k_0) \right) \Delta(k_0) T \right] \end{aligned} \quad (6.7)$$

It is convenient to choose k_0 as a macroscopic scale in a regime where there are no strong RG effects any more.

Furthermore, let us assume for a moment that at k_0 the cosmological constant is tiny, $\bar{\lambda}_{k_0} \approx 0$, so that $\langle g_{\mu\nu} \rangle_{k_0}$ is an approximately flat metric. In this case the trace in eq. (6.7) is easily evaluated in a plane wave basis:

$$P(T) = \int \frac{d^4 p}{(2\pi)^4} \exp \left[-p^2 F(p^2) T \right] \quad (6.8)$$

The T -dependence of (6.8) determines the fractal dimensionality of spacetime via (6.1). In the limits $T \rightarrow \infty$ and $T \rightarrow 0$ where the random walks probe very large and small distances, respectively, we obtain the dimensionalities corresponding to the largest and smallest length scales possible. The limits $T \rightarrow \infty$ and $T \rightarrow 0$ of $P(T)$ are determined by the behavior of $F(p^2) \equiv \bar{\lambda}(k = \sqrt{p^2})/\bar{\lambda}_{k_0}$ for $p^2 \rightarrow 0$ and $p^2 \rightarrow \infty$, respectively.

For a RG trajectory where the renormalization effects stop below some threshold we have $F(p^2 \rightarrow 0) = 1$. In this case (6.8) yields $P(T) \propto 1/T^2$, and we conclude that the macroscopic spectral dimension is $\mathcal{D}_s = 4$.

In the fixed point regime we have $\bar{\lambda}_k \propto k^2$, and therefore $F(p^2) \propto p^2$. As a result, the exponent in (6.8) is proportional to p^4 now. This implies the $T \rightarrow 0$ -behavior $P(T) \propto 1/T$. It corresponds to the spectral dimension $\mathcal{D}_s = 2$.

This result holds for all RG trajectories since only the fixed point properties were used. In particular it is independent of $\bar{\lambda}_{k_0}$ on macroscopic scales. Indeed, the above

assumption that $\langle g_{\mu\nu} \rangle_{k_0}$ is flat was not necessary for obtaining $\mathcal{D}_s = 2$. This follows from the fact that even for a curved metric the spectral sum (6.7) can be represented by an Euler-Mac Laurin series which always implies (6.8) as the leading term for $T \rightarrow 0$.

Thus we may conclude that on very large and very small length scales the spectral dimensions of the QEG spacetimes are

$$\begin{aligned}\mathcal{D}_s(T \rightarrow \infty) &= 4 \\ \mathcal{D}_s(T \rightarrow 0) &= 2\end{aligned}\tag{6.9}$$

The dimensionality of the fractal at sub-Planckian distances is found to be 2 again, as in the first argument based upon η_N . Remarkably, the equality of $4 + \eta$ and \mathcal{D}_s is a special feature of 4 classical dimensions. Generalizing for d classical dimensions, the fixed point running of Newton's constant becomes $G_k \propto k^{2-d}$ with a dimension-dependent exponent, while $\bar{\lambda}_k \propto k^2$ continues to have a quadratic k -dependence. As a result, the $\tilde{\mathcal{G}}(k)$ of eq. (5.6) is proportional to $1/p^d$ in general so that, for any d , the 2-dimensional looking graviton propagator (5.8) is obtained. (This is equivalent to saying that $\eta = 2 - d$, or $d + \eta = 2$, for arbitrary d .)

On the other hand, the impact of the RG effects on the diffusion process is to replace the operator Δ by Δ^2 , for any d , since the cosmological constant always runs quadratically. Hence, in the fixed point regime, eq. (6.8) becomes $P(T) \propto \int d^d p \exp[-p^4 T] \propto T^{-d/4}$. This T -dependence implies the spectral dimension

$$\mathcal{D}_s(d) = d/2\tag{6.10}$$

This value coincides with $d + \eta$ if, and only if, $d = 4$. It is an intriguing speculation that this could have something to do with the observed macroscopic dimensionality of spacetime.

For the sake of clarity and to be as explicit as possible we described the computation of \mathcal{D}_s within the Einstein-Hilbert truncation. However, it is easy to see [39] that the only nontrivial ingredient of this computation, the scaling behavior $\Delta(k) \propto k^2$, is in fact

an exact consequence of asymptotic safety. If the fixed point exists, simple dimensional analysis implies $\Delta(k) \propto k^2$ at the un-truncated level, and this in turn gives rise to (6.10). If QEG is asymptotically safe, $\mathcal{D}_s = 2$ at sub-Planckian distances is an *exact* nonperturbative result for all of its spacetimes.

It is interesting to compare the result (6.9) to the spectral dimensions which were recently obtained by Monte Carlo simulations of the causal dynamical triangulation model of quantum gravity [43]:

$$\begin{aligned}\mathcal{D}_s(T \rightarrow \infty) &= 4.02 \pm 0.1 \\ \mathcal{D}_s(T \rightarrow 0) &= 1.80 \pm 0.25\end{aligned}\tag{6.11}$$

These figures, too, suggest that the long-distance and short-distance spectral dimension should be 4 and 2, respectively. The dimensional reduction from 4 to 2 dimensions is a highly nontrivial dynamical phenomenon which seems to occur in both QEG and the discrete triangulation model. We find it quite remarkable that the discrete and the continuum approach lead to essentially identical conclusions in this respect. This could be a first hint indicating that the discrete model and QEG in the average action formulation describe the same physics.

7 Summary

In the first part of this article we reviewed the asymptotic safety scenario of quantum gravity, and the evidence supporting it coming from the average action approach. We explained why it is indeed rather likely that 4-dimensional Quantum Einstein Gravity can be defined (“renormalized”) nonperturbatively along the lines of asymptotic safety. The conclusion is that it seems quite possible to construct a quantum field theory of the spacetime metric which is not only an effective, but rather a fundamental one and which is mathematically consistent and predictive on the smallest possible length scales even. If so, it is not necessary to leave the realm of quantum field theory in order to construct a

satisfactory quantum gravity. This is at variance with the basic credo of string theory, for instance, which is also claimed to provide a consistent gravity theory. Here a very high price has to be paid for curing the problems of perturbative gravity, however: one has to live with infinitely many (unobserved) matter fields.

In the second part of this review we described the spacetime structure in nonperturbative, asymptotically safe gravity. The general picture of the QEG spacetimes which emerged is as follows. At sub-Planckian distances spacetime is a fractal of dimensionality $\mathcal{D}_s = 4 + \eta = 2$. It can be thought of as a self-similar hierarchy of superimposed Riemannian manifolds of any curvature. As one considers larger length scales where the RG running of the gravitational parameters comes to a halt, the “ripples” in the spacetime gradually disappear and the structure of a classical 4-dimensional manifold is recovered.

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